On the Impulsive Motion of Flat Plate in a Generalized Second Grade Fluid

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We investigate the effect of a transverse magnetic field on the unsteady flow of a generalized second grade fluid through a porous medium past an infinite flat plate. Using fractional partial differential equations, we are able to describe the velocity and stress fields of the flow. We also obtain exact analytic solutions of these differential equations in terms of the Fox's H-function.

Key words: Generalized Second Grade Fluid; Integral Transforms; Fractional Calculus; Wright Function; Fox's H-function.

1. Introduction

Non-Newtonian fluids are now considered to play a more important and appropriate role in technological applications in comparison with Newtonian fluids. A large class of real fluids does not exhibit the linear relationship between stress and the rate of strain. Analysis of the behavior of the fluid motion of non-Newtonian fluids has revealed that it tends to be much more complicated and subtle in comparison with that of Newtonian fluids [1]. The study of flow for an electrically conducting fluid along a flat plate under the influence of a magnetic field has been extensively investigated. Work in this area has many applications in engineering problems such as magnetohydrodynamics (MHD) generators, plasma studies, nuclear reactors, geothermal energy extraction, and the boundary layer control in the field of aerodynamics [2].

There are very few cases in which the exact analytic solution of the Navier-Stokes equation can be obtained. Exact solutions are even more rare if the constitutive equations for a viscoelastic fluid are considered. Although there are many models used to describe the viscoelastic behavior of fluids, the fluid of differential type has received special attention [3]. Rheological constitutive equations with fractional derivatives have long played an important role in the description of the properties of polymer solutions and melts. These equations are derived from well-known models (e.g., the Maxwell model) by substituting ordinary derivatives of the first, second, and higher orders with fractional derivatives of noninteger orders. In this way, the order of the derivative is related to a material parameter that can be associated with the degree of conversion as for the sol-gel transition. In some cases, it has been shown that constitutive equations employing fractional derivatives are linked to molecular theories [4]. At least the modified viscoelastic models are appropriate to describe the behaviors of xanthan gum and Sesbania gels [5]. In recent years, the flow of fluids through porous media has become an important topic because of the recovery of crude oil from the pores of reservoir rocks. In this case, the gross effect is represented by Darcy's law. The Rayleigh-Taylor instability of two superposed couple-stress fluids of uniform densities in a porous medium in the presence of a uniform horizontal magnetic field has recently been studied by Sunil et al. [6]. In the past few years, several simple flow problems associated with classical hydrodynamics have received new attention within the more general context of magnetohydrodynamics (MHD). The study of the motion of non-Newtonian fluids in the absence, as well as in the presence, of a magnetic field has applications in many areas, including the handling of biological fluids and the flow of nuclear fuel slurries, liquid metals and alloys, plasma, mercury amalgams, and blood. Another important field of application is electromagnetic propulsion. Basically, an electromagnetic propulsion system consists of a power source (such as a nuclear reactor), plasma, and a tube through which the plasma is accelerated by electromagnetic forces. The study of such systems, which is closely associated with magnetochemistry, requires a complete understanding of the equation of state and transport properties such as diffusion, the shear stress-shear rate relationship, thermal conductivity, electrical conductivity, and radiation. Some of these properties will undoubtedly be influenced by the presence of an external magnetic field that sets the plasma in hydrodynamic motion [7].

There has been a renewed interest in studying magnetohydrodynamic (MHD) flow and heat transfer in porous and non-porous media because of the effect of magnetic fields on the performance of many systems. For example, Raptis et al. [8] have analyzed hydromagnetic free convection flow through a porous medium between two parallel plates. Aldoss et al. [9] have studied mixed convection flow from a vertical plate embedded in a porous medium in the presence of a magnetic field. Chamkha [10] has considered MHD free convection flow from a vertical plate embedded in a thermally stratified porous medium with Hall effects.

Chakravarty [11] has studied the laminar convection flow of an incompressible electrically conducting second order viscoelastic fluid in a porous medium in the presence of a uniform transverse magnetic field. In this paper, the flow of an electrically conducting second grade fluid is investigated near a wall that is suddenly set in motion in the presence of a uniform magnetic field and through a porous medium. In our work, the velocity and stress fields of the flow are described by fractional partial differential equations. Exact analytic solutions of these differential equations are obtained by using the discrete Laplace transform of the sequential fractional derivatives in terms of the Fox's H-function.

2. Basic Equations

We begin by recalling the definition of the constitutive relationship of a second grade fluid. The Cauchy stress tensor T for a second grade fluid has the following form [1]:

$$T = -p\boldsymbol{I} + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{1}$$

where $-p$ is the pressure, $\boldsymbol{I}$ is the unit vector, α_1 and α_2 are the normal stress moduli, A_1 and A_2 are the kinematical tensors defined through

$$A_1 = \mathrm{grad}V + (\mathrm{grad}V)^T, \tag{2}$$

$$A_2 = \frac{\partial A_1}{\partial t} + A_1(\mathrm{grad}V) + (\mathrm{grad}V)^T A_1, \tag{3}$$

where $\partial/\partial t$ denotes the material time derivative, V denotes the velocity. The following restrictions on the signs of the material moduli hold:

$$\mu \geq 0, \quad \alpha_1 \geq 0, \quad \alpha_1 + \alpha_2 = 0. \tag{4}$$

Fluids satisfying the above conditions are known as second grade fluids in the literature. Generally, the constitutive relationship of generalized second grade fluids also has the form (1), but A_2 is defined as follows [3]:

$$A_2 = D_t^\beta A_1 + A_1(\mathrm{grad}V) + (\mathrm{grad}V)^T A_1, \tag{5}$$

where D_t^β is Caputo fractional calculus operator and may be defined as [12]:

$$D_t^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-\tau)^{-\beta} f'(\tau)\mathrm{d}\tau, \quad 0 < \beta < 1, \tag{6}$$

where Γ is the Gamma function, D_t^β denotes the material time derivative of fractional order and f' represents the ordinary derivative. When $\beta = 0$ and $\alpha_1 = 0$, the constitutive relationship describes the classical viscous Newtonian fluid.

The simplest unsteady flow is that which results because of the impulsive motion of a flat plate, in its own plane, in an infinite mass of fluid that is otherwise at rest. This flow was first studied by Stokes and is known in the literature as *Stokes' first problem*. Ahmadi and Manvi [13] have derived a general equation of motion for flow through a porous medium. The porous material containing the fluid is in fact a non-homogeneous medium. For the sake of analysis, it is possible to replace it with a homogeneous fluid, which has dynamical properties equivalent to the local averages of the original non-homogeneous medium.

For the unsteady flow of a viscous incompressible fluid with constant properties, the complete hydromagnetic system can be reduced to two equations involving the velocity, the pressure, and the magnetic field (i.e., the modified Navier-Stokes equation and the induction equation along with the solenoidal conditions on the two vector equations). If the displacement current is neglected, then Maxwell's equation becomes:

$$\nabla \times E = -\eta \frac{\partial H}{\partial t}, \quad \nabla \times H = 4\pi J, \quad J = \sigma(E + \eta V \times H), \tag{7}$$

the modified Navier-Stokes equation becomes

$$\rho \frac{\mathrm{D}V}{\mathrm{D}t} = \nabla \cdot T + \eta J \times H, \tag{8}$$

and the solenoidal conditions on the two vectors become

$$\nabla \cdot V = 0; \quad \nabla \cdot H = 0, \tag{9}$$

where ρ, μ, η, σ are the density, viscosity, permeability of the magnetic field and the electric conductivity of the fluid respectively. In the above equations $H,\ E$ are the magnetic and electric intensities, J is the current density and $\mathrm{D}/\mathrm{D}t$ is the material derivative.

In the present problem, following [14], it is assumed that no applied or polarization voltages exist (i.e., $E = 0$). This corresponds to the case where no energy is being added or extracted from the fluid by electrical means. The magnetic effects are confined to retarding the flow and dissipating energy of motion into heat. Since the plate is infinite, the physical variables will depend on y and t only. In general, the electrical current flowing in the fluid gives rise to an induced magnetic field that distorts the applied magnetic field, which would exist if the fluid were an electrical insulator. However, since the viscous boundary layer is thin and thermally ionized air is at best a poor electrical conductor, it is permissible to neglect the induced field compared to the applied field acting along the $y-$axis. Now we consider the flow of a generalized second grade viscoelastic fluid near a flat plate that is suddenly moved in its own plane with a constant velocity U.

By selecting the x-axis along the wall in the direction of motion and the y-axis perpendicular to the wall and assuming that the side wall effects are neglected (i.e., the wall is infinitely long), we seek a solution for the velocity field in the form $\underline{\vee} = u(y,t)\imath$, where u is the velocity in the x coordinate direction and $\imath$ is the unit vector in the x direction.

The fluid is set into motion through the action of the stress at the plate. The stress is given by (1), (2), (4), and (5). By substituting $u(y,t)$ in these equations, we obtain

$$T_{xy} = \mu \frac{\partial u}{\partial y} + \alpha_1 D_t^\beta \frac{\partial u}{\partial y}, \tag{10}$$

and $T_{xx} = T_{yy} = T_{zz} = T_{yz} = 0$, where $T_{xy} = T_{yx}$. By inserting the stress components and the velocity in (8) and by using Maxwell's equations, we obtain [15]

$$\rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2} + \alpha_1 D_t^\beta \frac{\partial^2 u}{\partial y^2} - \sigma B_0^2 u - \frac{\mu}{K} u, \tag{11}$$

where B_0 is the external magnetic field and K is the permeability of the isotropic porous medium.

For this problem, the boundary and initial conditions are as follows:

$$\begin{aligned} u(y,0) &= 0 \quad \text{for} \quad y > 0, \\ u(0,t) &= U \quad \text{for} \quad t > 0, \\ u(y,t) &= 0 \quad \text{at} \quad y \to \infty, \end{aligned} \tag{12}$$

3. Required Integral Transforms and Special Functions

The function of the Wright type is defined by the series expansion:

$$W(z;\alpha,\gamma) = \sum_{k=0}^{\infty} \frac{z^k}{k!\Gamma(\alpha k + \gamma)}. \tag{13}$$

Using the definition of the Wright function and the series expression of error function and exponential, we can easily prove the following:

$$W\left(-z; \frac{-1}{2}, 1\right) = \operatorname{erfc}\left(-\frac{z}{2}\right), \tag{14}$$

$$W\left(-z; -\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{\sqrt{\pi}} \exp\left(-\frac{z^2}{4}\right). \tag{15}$$

The formula for the Laplace transform of the Caputo fractional derivative is as follows:

$$\begin{aligned} &\int_0^\infty e^{-st} D_t^\beta f(t)\mathrm{d}t \\ &= s^\beta \tilde{F}(s) - \sum_{k=0}^{n-1} s^{\beta-k-1} D_t^k f(t)|_{t=0}, \\ &\quad (n-1 < \alpha < n). \end{aligned} \tag{16}$$

Because this formula involves the values of the function $f(t)$ and its derivatives at $t = 0$, for which a certain physical interpretation exists, we can expect that it can be useful in solving applied problems leading to linear fractional differential equations with constant coefficients and accompanying initial conditions in traditional form.

The Fox function, also referred as the Fox's H-function, generalizes the Mellin-Barnes function. The importance of the Fox function lies in the fact that it includes nearly all special functions occurring in applied mathematics and statistics as special cases. In 1961, Fox defined the H-function as the Mellin-Barnes type path integral [16]:

$$H_{p\ q}^{m\ n}\left[\sigma|_{(b_k,B_k)_1^q}^{(a_k,A_k)_1^p}\right]=\frac{1}{2\pi i}\int_\ell \frac{\Pi_{k=1}^m(b_k-B_ks)\Pi_{j=1}^n\Gamma(1-a_j+sA_j)}{\Pi_{k=m+1}^q\Gamma(1-b_k+sB_k)\Pi_{j=n+1}^p\Gamma(a_j-sA_j)}\sigma^s\mathrm{d}s, \tag{17}$$

where ℓ is a suitable contour in C, the orders (m,n,p,q) are integers $0\le m\le q, 0\le n\le p$ and the parameters $a_j\in R, A_j>0$, $j=1,2,\dots p$, $b_k\in R, B_k>0$, $k=1,2,..q$ are such that $A_j(b_k+i)\ne B_k(a_j-i-1)$, $i=0,1,2\dots$

$$H_{p\ q+1}^{1\ p}\left[-\sigma|_{(0,1),(1-b_1,B_1),\dots(1-b_q,B_q)}^{(1-a_1,A_1)\dots(1-a_p,A_p)}\right]=\sum_{k=0}^{\infty}\frac{\Gamma(a_1+A_1k)\dots\Gamma(a_p+A_pk)}{k!\Gamma(b_1+B_1k)\dots\Gamma(b_q+B_qk)}\sigma^k. \tag{18}$$

4. Velocity Field

Let us introduce the dimensionless variables

$$u^*=\frac{u}{U},\quad y^*=\frac{yU\rho}{\mu},\quad t^*=\frac{tU^2\rho}{\mu}, \tag{19}$$

in which U and $\mu/U^2\rho$ denote the characteristics velocity and time, respectively. The dimensionless govern equation (11) and its initial and boundary conditions (12) (if the asterisks are dropped) can be written as:

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial y^2}+\eta D_t^\beta\frac{\partial^2 u}{\partial y^2}-h\,u, \tag{20}$$

$$\begin{aligned}&u(y,0)=0 \quad \text{for} \quad y>0,\\ &u(0,t)=1 \quad \text{for} \quad t>0,\\ &u(y,t)=0 \quad \text{at} \quad y\to\infty,\end{aligned} \tag{21}$$

where η is a dimensionless viscoelasic parameter and $h=\sigma {B_0}^2\mu/\rho^2U^2+\mu^2/K\rho^2U^2$.

The Laplace integral transform, as a rule, is used for the analysis of various mechanical systems whose behavior is described by linear viscoelastic models with fractional derivatives [17]. Using the Laplace transform of (20) and (21) we arrive at

$$\frac{\mathrm{d}^2\tilde{u}}{\mathrm{d}y^2}-\frac{s+h}{1+\eta s^\beta}\tilde{u}=0, \tag{22}$$

and the boundary conditions will be

$$\begin{aligned}&\tilde{u}(y,s)=\tfrac{1}{s} \quad \text{at} \quad y=0,\\ &\tilde{u}(\infty,s)=0.\end{aligned} \tag{23}$$

Solving the upper equations yields:

$$\tilde{u}(y,s)=\frac{1}{s}\exp\left(-y\left(\frac{s+h}{1+\eta s^\beta}\right)^{\frac{1}{2}}\right). \tag{24}$$

In order to avoid the burdensome calculations of residues and contour integrals, we will apply the discrete inverse Laplace transform method to give the velocity distribution.

Firstly, we rewrite (24) as a series form:

$$\tilde{u}(y,s)=\sum_{k=0}^{\infty}\frac{(-y\eta^{-\frac{1}{2}})^k}{k!}\sum_{m=0}^{\infty}\frac{(-h)^m}{m!}\sum_{n=0}^{\infty}\frac{(-\eta)^{-n}\Gamma\left(n+\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}\right)}{n!\Gamma\left(\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)}\frac{1}{s^{(\beta-1)\frac{k}{2}+\beta n+m+1}}. \tag{25}$$

Applying on (25) the inverse Laplace transform, we obtain

$$\begin{aligned}u(y,t)&=\sum_{k=0}^{\infty}\frac{(-y\eta^{-\frac{1}{2}})^k}{k!}\sum_{m=0}^{\infty}\frac{(-h)^m}{m!}\sum_{n=0}^{\infty}\frac{(-\eta)^{-n}\Gamma\left(n+\frac{k}{2}\right)\Gamma\left(m-\frac{k}{2}\right)}{n!\Gamma\left(\frac{k}{2}\right)\Gamma\left(-\frac{k}{2}\right)\Gamma\left((\beta-1)\frac{k}{2}+\beta n+m+1\right)}t^{(\beta-1)k/2+\beta n+m}\\ &=\sum_{k=0}^{\infty}\frac{(-y\eta^{-\frac{1}{2}})^k}{k!}\sum_{m=0}^{\infty}\frac{(-h)^m}{m!}t^{(\beta-1)\frac{k}{2}+m}H_{2\ 4}^{1\ 2}\left[\eta^{-1}t^\beta|_{(0,1),(1-\frac{k}{2},0),(1+\frac{k}{2},0),(\frac{(1-\beta)k}{2}-m,\beta)}^{(1-\frac{k}{2},1),(1-m+\frac{k}{2},0)}\right],\end{aligned} \tag{26}$$

in which $H_{p\ q}^{m\ n}(z)$ denotes the H-function.

When $h \to 0$, (26) tends to the same obtained in [11]:

$$u(y,t) = \sum_{k=0}^{\infty} \frac{(-y\eta^{-\frac{1}{2}})^k}{k!} t^{(\beta-1)k/2} \cdot H^{1\,1}_{1\,3}\left[\eta^{-1}t^{\beta}\Big|^{(1-\frac{k}{2},1)}_{(0,1),(1-\frac{k}{2},0),((1-\beta)\frac{k}{2},\beta)}\right]. \tag{27}$$

Now let us setting $\beta = 0$, (27) can be simplified as:

$$\begin{aligned} u(y,t) &= \sum_{k=0}^{\infty} \frac{(-y)^k t^{-\frac{k}{2}}}{k!\Gamma\left(1-\frac{k}{2}\right)}\left(1+\eta\right)^{-\frac{k}{2}} \\ &= W\left(\frac{-y}{\sqrt{t(1+\eta)}};-\frac{1}{2},1\right) \\ &= \operatorname{erfc}\left(\frac{y}{2\sqrt{t(1+\eta)}}\right). \end{aligned} \tag{28}$$

If we set $\beta = 0$, in (26), it will be simplified as:

$$u(y,t) = \sum_{k=0}^{\infty} \frac{\left(\frac{-y}{\sqrt{t(1+\eta)}}\right)^k}{k!} \cdot H^{1\,1}_{1\,3}\left[ht\Big|^{(1+\frac{k}{2},1)}_{(0,1),(1+\frac{k}{2},0),(\frac{k}{2},1)}\right]. \tag{29}$$

Setting $\alpha_1 = 0$ in (29), we obtain

$$u(y,t) = \sum_{k=0}^{\infty} \frac{\left(\frac{-y}{\sqrt{t}}\right)^k}{k!} H^{1\,1}_{1\,3}\left[ht\Big|^{(1+\frac{k}{2},1)}_{(0,1),(1+\frac{k}{2},0),(\frac{k}{2},1)}\right]. \tag{30}$$

Finally if we put $h = 0$ in (30) then we obtain

$$u(y,t) = W\left(-\frac{y}{\sqrt{t}};-\frac{1}{2},1\right) = \operatorname{erfc}\left(-\frac{y}{2\sqrt{t}}\right), \tag{31}$$

which is reduces to the dimensionless classical Rayleigh's analogy solution of Newtonian fluid. It is obvious that the result of this paper includes special case the classical Newtonian fluid.

If we set $\beta = 1$, (26) can be simplified as:

$$u(y,t) = \sum_{k=0}^{\infty} \frac{(-y\eta^{-\frac{1}{2}})^k}{k!} \sum_{m=0}^{\infty} \frac{(-h)^m}{m!} \cdot H^{1\,2}_{2\,4}\left[\eta^{-1}t\Big|^{(1-\frac{k}{2},1),(1-m+\frac{k}{2},0)}_{(0,1),(1-\frac{k}{2},0),(1+\frac{k}{2},0),(-m,1)}\right]. \tag{32}$$

This is the solution of velocity for ordinary second grade fluid under the effect of the magnetic field.

5. Stress Field

Because the fluid is set into motion through the action of the stress at the plate under the effect of the magnetic field, the stress field should be calculated. From (10) the dimensionless stress can be represented as:

$$F(y,t) = \frac{\partial u}{\partial y} + \eta D_t^{\beta}\left(\frac{\partial u}{\partial y}\right), \tag{33}$$

where $F(y,t) = \frac{T_{xy}}{\rho U^2}$. The Laplace transform of (33) is:

$$\tilde{F}(y,s) = \frac{\mathrm{d}\tilde{u}}{\mathrm{d}y} + \eta s^{\beta}\frac{\mathrm{d}\tilde{u}}{\mathrm{d}y}. \tag{34}$$

Substituting (24) into (34), we have

$$\tilde{F}(y,s) = -\frac{1}{s}\left((1+\eta s^{\beta})(s-h)\right)^{\frac{1}{2}} \exp\left(-y\left(\frac{s+h}{1+\eta s^{\beta}}\right)^{\frac{1}{2}}\right). \tag{35}$$

We apply the discrete inverse Laplace transform method again to give the stress distribution. Firstly, we rewrite (35) as a series form:

$$\tilde{F}(y,s) = -\sum_{k=0}^{\infty} \frac{(-y)^k \eta^{\frac{1-k}{2}}}{k!} \sum_{m=0}^{\infty} \frac{(-h)^m \Gamma\left(m-\frac{k+1}{2}\right)}{m!\Gamma\left(-\frac{k+1}{2}\right)} \sum_{n=0}^{\infty} \frac{(-\eta^{-1})^n \Gamma\left(\frac{k-1}{2}+n\right)}{n!\Gamma\left(\frac{k-1}{2}\right)} \frac{1}{s^{(\beta-1)(k-1)/2+m+n\beta}}. \tag{36}$$

Applying the discrete inverse Laplace transforms with (36), we obtain

$$\begin{aligned} F(y,t) &= -\sum_{k=0}^{\infty}\frac{(-y)^k\eta^{\frac{1-k}{2}}}{k!}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-\eta^{-1})^n\Gamma\Big(n+\dfrac{k-1}{2}\Big)\Gamma\Big(m-\dfrac{k+1}{2}\Big)}{n!\Gamma\Big(\dfrac{k-1}{2}\Big)\Gamma\Big(-\dfrac{k+1}{2}\Big)}\frac{t^{(\beta-1)(k-1)/2+m+n\beta-1}}{\Gamma\Big((\beta-1)(k-1)/2+m+n\beta\Big)} \\ &= -\sum_{k=0}^{\infty}\frac{(-y)^k\eta^{\frac{1-k}{2}}}{k!}\sum_{m=0}^{\infty}\frac{(-h)^m}{m!}t^{(\beta-1)(k-1)/2+m-1}H^{1\,2}_{2\,4}\left[\eta^{-1}t^{\beta}\Big|^{(\frac{3+k}{2}-m,0),(\frac{3-k}{2},1)}_{(0,1),(\frac{3-k}{2},0),(\frac{3+k}{2},0),(1+\frac{(1-\beta)(k-1)}{2}-m,\beta)}\right]. \end{aligned} \tag{37}$$

If one sets $h=0$, (37) can be simplified as:

$$F(y,t) = -\sum_{k=0}^{\infty}\frac{(-y)^k\eta^{\frac{1-k}{2}}}{k!}t^{(\beta-1)(k-1)/2-1}H^{1\,1}_{1\,3}\left[\eta^{-1}t^{\beta}\Big|^{(\frac{3-k}{2},1)}_{(0,1),(\frac{3-k}{2},0),(1+\frac{(1-\beta)(k-1)}{2},\beta)}\right], \tag{38}$$

which is the same result obtained by Tan [3]. Now let us set $\beta=0$ in (37) thus:

$$\begin{aligned} F(y,t) &= -\sqrt{\frac{1+\eta}{t}}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{-y}{\sqrt{t(\eta+1)}}\Big)^k\sum_{m=0}^{\infty}\frac{\Gamma\Big(m-\dfrac{k+1}{2}\Big)}{m!\Gamma\Big(-\dfrac{1+k}{2}\Big)\Gamma\Big(m-\dfrac{k-1}{2}\Big)}(-ht)^m \\ &= -\sqrt{\frac{1+\eta}{t}}\sum_{k=0}^{\infty}\frac{1}{k!}\Big(\frac{-y}{\sqrt{t(1+\eta)}}\Big)^k H^{1\,1}_{1\,3}\left[ht\Big|^{(\frac{3+k}{2},1)}_{(0,1),(\frac{3+k}{2},0),(\frac{k+1}{2},1)}\right]. \end{aligned} \tag{39}$$

If $h=0$ we obtain

$$F(y,t) = -\sqrt{\frac{1+\eta}{t}}W\Big(\frac{-y}{\sqrt{t(\eta+1)}};-\frac{1}{2},\frac{1}{2}\Big) = -\sqrt{\frac{1+\eta}{t\pi}}\exp\Big(\frac{-y^2}{4t(\eta+1)}\Big). \tag{40}$$

If we put $\alpha_1=0$ in (39) we obtain

$$F(y,t) = -\sqrt{\frac{1}{t}}\sum_{m=0}^{\infty}\frac{1}{k!}\Big(\frac{-y}{\sqrt{t}}\Big)^k H^{1\,1}_{1\,3}\left[ht\Big|^{(\frac{3+k}{2},1)}_{(0,1),(\frac{3+k}{2},0),(\frac{k+1}{2},1)}\right], \tag{41}$$

putting $h=0$ we arrive at

$$F(y,t) = -t^{-\frac{1}{2}}W\Big(\frac{-y}{\sqrt{t}};-\frac{1}{2},\frac{1}{2}\Big) = -(t\pi)^{-\frac{1}{2}}\exp\Big(-\frac{y^2}{4t}\Big). \tag{42}$$

In order to obtain the value of the shear stress at the plate, we take $y=0$ in (35), to obtain the following expression:

$$\tilde{F}_p(s) = -\frac{1}{s}\Big((1+\eta s^{\beta})(s+h)\Big)^{\frac{1}{2}} = -\eta^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-h)^m\Gamma\Big(m-\dfrac{1}{2}\Big)}{m!\Gamma\Big(-\dfrac{1}{2}\Big)}\sum_{n=0}^{\infty}\frac{(-\eta^{-1})^n\Gamma\Big(n-\dfrac{1}{2}\Big)}{n!\Gamma\Big(-\dfrac{1}{2}\Big)}\frac{1}{s^{\frac{1-\beta}{2}+m+n\beta}}. \tag{43}$$

Similarly we obtain the following formula to calculate the shear stress at the plate under the effect of the magnetic field.

$$\begin{aligned} F_p(t) &= -\eta^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-h)^m}{m!}\sum_{n=0}^{\infty}\frac{(-\eta^{-1})^n\Gamma\Big(m-\dfrac{1}{2}\Big)\Gamma\Big(n-\dfrac{1}{2}\Big)}{n!\Big(\Gamma\Big(-\dfrac{1}{2}\Big)\Big)^2}\frac{t^{m-\frac{1+\beta}{2}+n\beta}}{\Gamma\Big(\dfrac{1-\beta}{2}+m+n\beta\Big)} \\ &= -\eta^{\frac{1}{2}}\sum_{m=0}^{\infty}\frac{(-h)^m}{m!}t^{m-\frac{1+\beta}{2}}H^{1\,2}_{2\,4}\left[\eta^{-1}t^{\beta}\Big|^{(\frac{3}{2}-m,0),(\frac{3}{2},1)}_{(0,1),(\frac{3}{2},0),(\frac{3}{2},0),(\frac{1+\beta}{2}-m,\beta)}\right]. \end{aligned} \tag{44}$$

Setting $h = 0$ in (43) and (44) we arrive at

$$\tilde{F}_p(s) = -\eta^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-\eta^{-1})^n \Gamma\left(n-\frac{1}{2}\right)}{n!\Gamma\left(-\frac{1}{2}\right)} \frac{1}{s^{\frac{1-\beta}{2}+n\beta}}, \tag{45}$$

and

$$F_p(t) = -\eta^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-\eta^{-1})^n \Gamma\left(n-\frac{1}{2}\right)}{n!\Gamma\left(-\frac{1}{2}\right)} \frac{t^{-\frac{1+\beta}{2}+n\beta}}{\Gamma\left(\frac{1-\beta}{2}+n\beta\right)} = -\eta^{\frac{1}{2}} t^{-\frac{1+\beta}{2}} H^{1\,1}_{1\,3}\left[\eta^{-1}t^{\beta}\Big|^{(\frac{3}{2},1)}_{(0,1),(\frac{3}{2},0),(\frac{1+\beta}{2},\beta)}\right]. \tag{46}$$

Finally setting $\beta = 0$ we obtain:

$$F_p(t) = \sqrt{\frac{1+\eta}{t\pi}}. \tag{47}$$

6. The Relationship between Stress Field and Velocity Field

Substituting (24) into (35), we obtain

$$\tilde{F}(y,s) = -\sqrt{(s+h)(1+\eta s^{\beta})}\tilde{u}(s). \tag{48}$$

The Laplace inversion of equation (48) takes the form

$$F(y,t) = -\eta^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-h)^k}{k!} \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)(-\eta^{-1})^n}{n!\left(\Gamma\left(-\frac{1}{2}\right)\right)^2 \Gamma\left(n\beta+k-\frac{\beta+1}{2}\right)} \int_0^t (t-\tau)^{n\beta+k-\frac{\beta+1}{2}-1} u(y,\tau)\,\mathrm{d}\tau. \tag{49}$$

Using the definition of the fractional calculus, (49) can be written as:

$$F(y,t) = -\eta^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-h)^k}{k!} \sum_{n=0}^{\infty} \frac{\Gamma\left(n-\frac{1}{2}\right)\Gamma\left(k-\frac{1}{2}\right)(-\eta^{-1})^n}{n!\left(\Gamma\left(-\frac{1}{2}\right)\right)^2} D^{-n\beta-k+\frac{\beta+1}{2}} u(y,t). \tag{50}$$

The physical meaning of (49) and (50) is that the stress at a given point at any time depends on the time history of the velocity profile at that point, and that the time history can be depicted by the fractional calculus [5]. If we set $h = 0$, we obtain the same results given by Tan [3]. If one sets $\beta = 0,\ \alpha_1 = 0$ and $h = 0$, formula (50) can be simplified as $F(y,t) = D_t^{1/2} u(y,t)$. This is the dimensionless form of the result obtained by Bagley and Torvik [18].

7. Discussion

The velocity of distribution for the flow of a generalized second grade fluid for small electrical conductivity near an infinite plate in the presence of a transverse magnetic field is given by (26). If we put $\eta = 0$, we obtain the results for hydromagnetic Newtonian flow. If we take $h = 0$, we obtain the results as studied by Tan [3]. The fact that the drag increases with the increase in the magnetic field is attributed to the following physical considerations. Because of the motion of the fluid in the magnetic field, an associated electrical field is reduced, which, according to Ohm's law, sets up electrical currents in the fluid if the latter is a conductor. The interaction of these currents with the magnetic field then produces a body force that must be included in the Navier-Stokes equa-

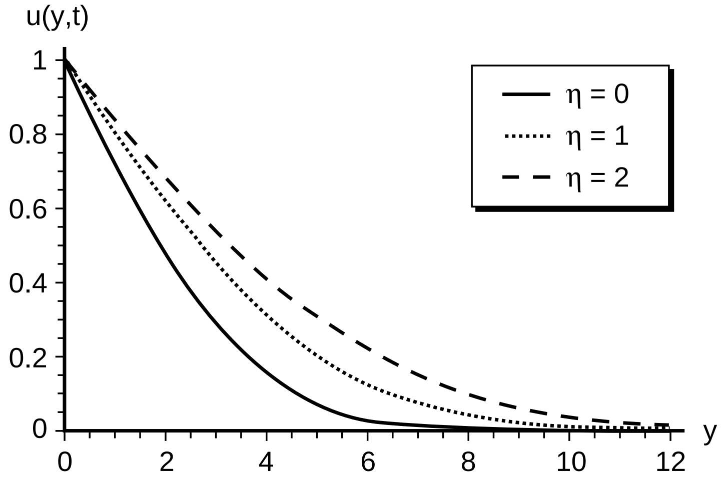

Fig. 1. $u(y,t)$ for various values of β with $t = 2$.

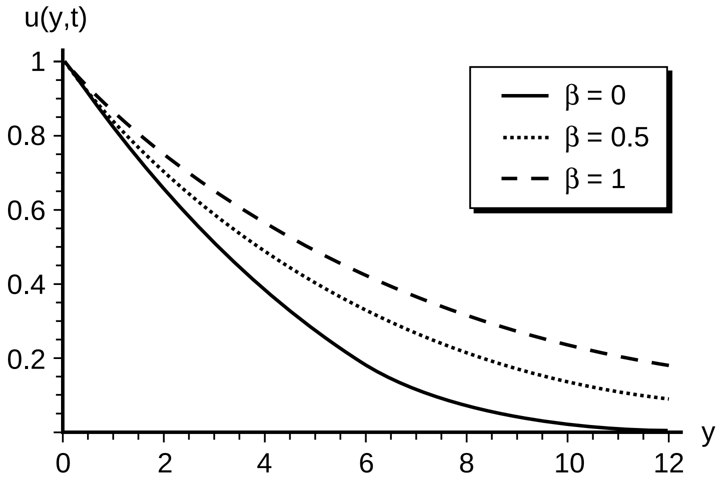

Fig. 2. $u(y,t)$ for various values of η with $t = 2$.

tion for the motion of the fluid. The effect of this force is inhibited in the motion of the fluid across the lines of force. The natural tendency of the fluid to move along the plate is opposed, resulting in an increase in drag. Figure 1 shows the effect of fractional derivative β on the velocity. An increase of the fractional derivative β is shown to decrease the velocity.

The velocities for different values of the viscoelastic parameter η are plotted in Figure 2. It is obvious that the effect of the viscoelastic parameter results in increasing the velocity. The effect of the permeability parameter K is to increase the velocity. Hence, we conclude that a porous medium decreases velocity. It has demonstrated a close analogy between a viscoelastic medium and an electrically conducting fluid containing a magnetic field, as mentioned by Ogilivie and Proctor [19]. We hope that our results will be used to validate and interpret more complicated types of flow and rheological models.

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